Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Mathematics and Mechanics 70 (2006) 924–933
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Non-linear near-resonance oscillations of an elastic incompressible layer☆

A.G. Kulikovskii, E.I. Sveshnikova

Moscow, Russia

Received 11 April 2006

Abstract

Plane one-dimensional waves of small amplitude, propagating transverse to an incompressible elastic layer and reflected successively from its boundaries, are considered. The oscillations are caused by small periodic (or close to periodic) external action on one of the layer boundaries, when the period of the external action is close to the period of natural oscillations of the layer. One of the boundaries of the elastic layer is fixed, while the other performs small specified two-dimensional motion in its plane. In such a near-resonance situation, non-linear effects occur which may build up over time. A system of equations is obtained which describes the slow change in the functions characterizing the oscillations of the medium in each period of the external action. It is assumed that all the quantities depend both on real time, any change of which in the approach considered is limited to one period, and on "slow" time, for which one period of real time serves as a small quantity. It is assumed that the evolution of the solution occurs when the slow time changes, while the role of real time is similar to the role of a spatial variable. This system of equations is obtained by the method of averaging over a period of the quantities representing nonlinear terms and the effect of the boundary conditions in the equations. It contains derivatives with respect to the real and slow times and also values of the functions characterizing the solution averaged over a period of the real time. If the averaged values are known, the equations have a hyperbolic form and their solutions can be both continuous and contain weak and strong discontinuities.

It was shown in Refs 1,2, when investigating slightly non-linear plane waves in elastic media, that in an incompressible medium only transverse waves, described by equations with a cubic non-linearity, can propagate. In the equations terms describing slight anisotropy of the medium were also taken into account,[2] and it was assumed that these terms are of the same order as the non-linear terms. The smallness of the non-linearity and the anisotropy means that a considerable time is required for effects related to these to manifest themselves. Thus, if ε represents the change in the quantities in a wave, then, for example, the time of breaking of a Riemann wave of finite length will be of the order of ε^{-2}. These effects can only be observed in limited volumes of a medium when the wave travels through the medium many times and is reflected from its boundaries. This occurs, for example, in the problem of near-resonance oscillations of an elastic layer, excited by periodic action on one of its boundaries.

Steady near-resonance oscillations have been investigated in detail in a gas (see, for example, Refs 3–8), situated in a tube for different conditions at its ends. Equations were obtained describing the process by which periodic oscillations of the gas become established.[6] Transverse oscillations in a layer of an isotropic elastic medium were investigated in

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 6, pp. 1031–1041, 2006.
E-mail address: kulik@mi.ras.ru (A.G. Kulikovskii).

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2007.01.009

the case when one component of the stresses produces the transverse oscillations and the average value of the stresses over a period is zero.[9] These oscillations, as mentioned, are described by equations with a cubic non-linearity, unlike the quadratic non-linearity in the case of the oscillations of a gas. Plane motions were considered in Ref. 10 for the oscillations of a layer of plasma in a magnetic field, orthogonal to the layer, with the particular assumption that the velocity of sound is identical with the Alfven velocity. This leads to complex resonance interaction of the transverse Alfven perturbations and the longitudinal acoustic waves.

For oscillations of a layer of a slightly anisotropic elastic incompressible medium, considered below, two types of waves propagate in each direction, the velocities of which differ by a quantity of the order of ε^2, determined by the effect of the non-linearity and the anisotropy. Hence, near-resonance conditions occur simultaneously for both types of waves. The presence of anisotropy of the medium makes it natural to investigate oscillations in which the medium performs arbitrary motions in planes orthogonal to the direction of wave propagation. For simplicity we will assume the medium to be incompressible. If the medium is compressible, but there are no resonances between the longitudinal and transverse perturbations, longitudinal perturbations will not develop to any great extent and we would expect that the transverse oscillations will be close to the oscillations of an incompressible medium considered previously.

Below we obtain equations which describe the development of oscillations when there are external actions, close to periodic, on the layer surface, i.e. the actions vary only slightly from period to period, and the length of the period itself may also change slowly. It is also permissible for the properties of the elastic medium to vary slowly with time. The equations obtained can be used both to investigate the oscillations which arise and vary when there are specified external actions, and also to determine the external actions required to maintain a previously specified form of the oscillations or their variations in accordance with a certain program. An example is considered in which the actions required to maintain oscillations of a specified form are calculated.

1. Formulation of the problem

Suppose an incompressible elastic layer of width L is situated between two parallel planes, orthogonal to a certain direction, taken as the axis $x_3 = x$ of a Cartesian Lagrange system of the initial state. We will consider plane one-dimensional transverse waves propagating in the direction of the x axis. The x_1 and x_2 axes lie in the plane parallel to the wave front.

The elastic medium is slightly non-linear, possesses slight anisotropy and can be specified by its elastic potential Φ in the form of an expansion in series in the small components of the tensor of the displacement gradients $\partial w_i/\partial x_j (i, j = 1, 2, 3)$, where w_i are the components of the displacement vector. Only the components $\partial w_i/\partial x_j = u_i(x, t)$ vary in plane waves. In an incompressible medium $u_3 = \partial w_3/\partial x = \text{const} = 0$ and hence $\Phi = \Phi(u_1, u_2)$. If in the expansion of the elastic potential Φ in series the variable components u_i of the deformation, which are assumed to be small, we can confine ourselves to the principal terms, that exhibit the non-linearity and anisotropy of the medium, then in the general case we can represent the function Φ in the form[1,2]

$$\Phi = f\frac{u_1^2 + u_2^2}{2} + g\frac{u_2^2 - u_1^2}{2} - \kappa\frac{(u_1^2 + u_2^2)^2}{4} \tag{1.1}$$

The coefficient f in the first term differs only slightly from the shear modulus μ and is proportional to the square of the velocity of small perturbations in a linear isotropic medium $c_0^2 = \rho_0$. The last term represents the non-linear properties of the medium, its coefficient κ is finite and can have any sign. The factor g on the anisotropic term is assumed to be small and positive. In order that the action of the effects of non-linearity and anisotropy should be of the same order, we must assume that $g \sim \varepsilon^2$ in expansion (1.1), where ε is the order of magnitude of u_i. The medium is assumed to be uniform and its density $\rho_0 = \text{const}$. We will henceforth assume that $\rho_0 = 1$, so that $c_0^2 = f$, which can always be achieved by an appropriate choice of the system of measurement units.

The differential equations of one-dimensional motions of an incompressible elastic medium in Lagrange variables can be represented by the hyperbolic system

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial u_i}\right), \quad \frac{\partial u_i}{\partial t} = \frac{\partial v_i}{\partial x}, \quad i = 1, 2 \tag{1.2}$$

Here $v_i = \partial w_i/\partial t$ are the components of the velocity vector.

It is useful to note that, for waves of small amplitude propagating in only one direction (for example, in the direction of the x axis), this system of four equations can be converted approximately, but without loss of accuracy, into a system of two equations[2,11] with potential Φ_1 of the same structure as (1.1), but with somewhat changed coefficients $f \to f_1, g \to g_1, \kappa \to \kappa_1$:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi_1}{\partial u_i}\right), \quad i = 1, 2$$
$$f_1 = \sqrt{f}, \quad g_1 = g/(2\sqrt{f}), \quad \kappa_1 = \kappa/(2\sqrt{f}) \tag{1.3}$$

In the linear approximation in a layer of an isotropic elastic medium between planes $x=-L$ and $x=0$, free from external actions, periodic natural perturbations can exist which propagate along the x axis with a velocity c_0 and a period of natural oscillations $T_0=2L/c_0$. As a result of anisotropy the velocity of linear oscillations will differ from c_0 by a quantity of the order of ε^2.

Suppose now that one of the boundaries ($x=-L$) is subjected to a small external periodic action in the form of a small periodic motion of the boundary of the layer in its plane. Plane transverse waves, excited in the layer, will travel in the direction of the x axis, and will be alternately reflected from one boundary of the layer to the other. We will first assume that the period of the external action is constant ($T=\text{const}$) and the properties of the medium do not change with time.

The following conditions will be satisfied on the boundaries: $v_i = 0$ when $x=0$ and $v_i = \psi_i(t)$ when $x=-L$, where the functions $\psi_i(t)$ are periodic with period $T \neq T_0$ and have small amplitude not less than an order of magnitude smaller than u_i. It will be seen later that if this amplitude is of the order of ε^3, the effect of the external actions may compensate the action of the non-linear terms and steady periodic oscillations of the medium will be possible.

We will now assume that the period of the external actions T is constant and close to the period of natural oscillations T_0, so that the quantity $a=2L/T=\text{const}$ differs only slightly from the velocity of natural oscillations c_0, where $a^2 - c_0^2 \sim \varepsilon^2$. With these assumptions, it is necessary to take into account non-linear terms in Eq. (1.2), while the solution in a single period can be constructed as the sum of linear waves propagating with a velocity a, and small corrections which arise from the action of non-linear terms, which, together with the action of the boundary conditions, lead to a slow change in the waves from period to period.

2. Conversion of the equations

The linear approximation. Instead of the elastic potential Φ we will introduce another function

$$F(u_1, u_2) = \Phi - a^2\frac{u_1^2 + u_2^2}{2} \tag{2.1}$$

Since a^2 differs from the coefficient f by a quantity of the order of ε^2, all terms of the new function F, including those quadratic in u_i, are of the order of ε^4. We will convert the first group of Eq. (1.2) so that all terms of the order of ε lie on the left-hand sides of the equations, and all terms of higher order of smallness (they are of the order of ε^3) are transferred to the right-hand sides. The right-hand sides can be written in terms of the new function F in the form

$$\frac{\partial v_i}{\partial t} - a^2\frac{\partial u_i}{\partial x} = b_i, \quad b_i = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_i}\right), \quad \frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial x} = 0; \quad i = 1, 2$$

The left-hand sides of this system of four equations can be reduced to the characteristic form

$$\frac{\partial v_i}{\partial t} - a\frac{\partial v_i}{\partial x} + a\left(\frac{\partial u_i}{\partial t} - a\frac{\partial u_i}{\partial x}\right) = b_i(u_k)$$
$$\frac{\partial v_i}{\partial t} + a\frac{\partial v_i}{\partial x} - a\left(\frac{\partial u_i}{\partial t} + a\frac{\partial u_i}{\partial x}\right) = b_i(u_k); \quad i, k = 1, 2 \tag{2.2}$$

For the left-hand sides of this system, the functions

$$w_i^+ = \upsilon_i - au_i, \quad w_i^- = \upsilon_i + au_i \tag{2.3}$$

are Riemann invariants which enable system (2.2) to be written in the form

$$\begin{aligned} \frac{dw_i^+}{dt} &= b_i \ \text{ along the characteristic } \ \frac{dx}{dt} = a \\ \frac{dw_i^-}{dt} &= b_i \ \text{ along the characteristic } \ \frac{dx}{dt} = -a \end{aligned} \tag{2.4}$$

where, on the right-hand sides of the equations u_k must be expressed in terms of $w_k^{\pm}$:

$$u_k = \frac{1}{2a}(w_k^- - w_k^+)$$

We will use the method of successive approximations to investigate the problem. We will take the solution of the linear system as the zeroth solution, when the right-hand sides (of the order of ε^3) can be neglected. These are travelling waves in the positive and negative directions of the x axis, in which the corresponding zeroth approximations of the Riemann invariants are preserved:

$$w_i^{+0} = \varphi_i(at - x), \quad w_i^{-0} = \vartheta_i(at + x)$$

and, correspondingly, for the components of the deformation and velocity the following equalities hold

$$u_i^0 = \frac{1}{2a}[\vartheta_i(at + x) - \varphi_i(at - x)], \quad \upsilon_i^0 = \frac{1}{2}[\vartheta_i(at + x) + \varphi_i(at - x)] \tag{2.5}$$

Since, by the conditions of the problem, the external action $\psi_i(t)$ has an amplitude that is much less than the quantity u_i, in the linear approximation considered on both boundaries we must take the zero boundary conditions $\upsilon_i^0 = 0$ when $x = 0$ and when $x = -L$.

The condition on the right boundary (where $x = 0$), gives

$$\vartheta_k = -\varphi_k \tag{2.6}$$

However, for the moment it is best to retain both functions $\varphi_i(at = x)$ and $\vartheta_i(at + x)$ in order that their notation can be indicated on the structure of their arguments. The second boundary condition for $x = -L$ indicates that the solution in the zeroth approximation is periodic with period $2L/a = T$.

3. Finding the next (non-linear) approximation

For an elastic potential of the form (1.1), the right-hand sides $b_i(u_k)$ of Eq. (2.4) have the form

$$b_i = g_i \frac{\partial u_i}{\partial x} - \kappa\left[(3u_i^2 + u_{3-i}^2)\frac{\partial u_i}{\partial x} + 2u_1u_2\frac{\partial u_{3-i}}{\partial x}\right]$$

The coefficients

$$g_1 = f - g - a^2, \quad g_2 = f + g - a^2$$

are of the order of ε^2, while the functions obtained $b_i \sim \varepsilon^3$. We will use the method of successive approximations to integrate Eq. (2.4). In view of the smallness of the functions b_i on the right-hand sides of Eq. (2.4) we will use the zeroth approximation (2.5) for u_i.

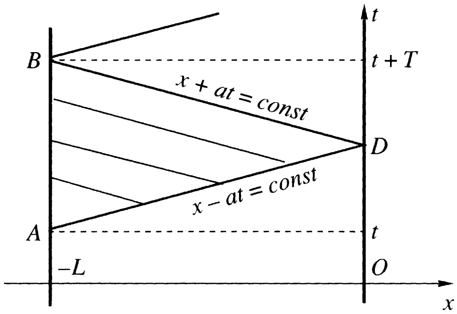

Fig. 1.

In the expressions for $b_i(\varphi_k, \vartheta_k)$, taking into account the form of the arguments of the functions φ_k and ϑ_k, the differentiation with respect to x can be replaced by differentiation with respect to t. Then b_i take the form

$$b_i = \frac{1}{2a^2}\left\{g_i - \frac{\kappa}{4a^2}[3(\varphi_i - \vartheta_i)^2 + (\varphi_{3-i} - \vartheta_{3-i})^2]\right\}\left(\frac{\partial\varphi_i}{\partial t} + \frac{\partial\vartheta_i}{\partial t}\right) + $$
$$+ (-1)^{3-i}\frac{\kappa}{4a^4}(\varphi_1 - \vartheta_1)(\varphi_2 - \vartheta_2)\left(\frac{\partial\varphi_{3-i}}{\partial t} + \frac{\partial\vartheta_{3-i}}{\partial t}\right) \tag{3.1}$$

We will integrate Eq. (2.4) with the right-hand sides (3.1) along their characteristics.

In Fig. 1, we show, in the characteristic (x, t) plane, the boundaries of the elastic layer $x = -L$ and $x = 0$ and the characteristic $x - at = \text{const}$ (AD), which moves to the right as the time increases, and the characteristics $x + at = \text{const}$ (DB), which moves to the left. The functions $\varphi_i(at - x)$ and their derivatives are constant on the characteristics (AD), moving to the right, while the functions $\vartheta_i(at + x)$ and their derivatives are constant on the characteristics (DB), moving to the left.

We will investigate how the state of the medium, i.e. u_i, v_i, and consequently, $w_i^{\pm}$, changes at a fixed point of space during a period T. We will choose as the observation point the left-hand boundary of the layer $x = -L$. We will take the state $w_i^{\pm}(-L, t)$ at the point A on the left-hand boundary as the initial state and, integrating Eq. (2.4), we will obtain these functions at the point B on the same boundary in terms of the period T, thereby satisfying the boundary conditions on the right-hand boundary of the layer. Integration of the first of the equations along its characteristic gives, at the point $D(0, t + T/2)$ on the right-hand boundary

$$w_i^+(D) \equiv w_i^+(0, t + T/2) = w_i^+(-L, t) + \int_t^{t+T/2} b_i dt$$

When evaluating the integrals of b_i (obtainable from each term) we take into account the fact that the functions $\varphi_i(at - x)$ and their derivatives are constant along the characteristic AD. Although the functions $\vartheta_i(at + x)$ vary along AD, their values can be assumed to be taken along their characteristics $(at + x = \text{const})$ from the section AB of the boundary $x = -L$. Hence, when integrating the equation for w_i^+ along AD, the functions ϑ_i must be integrated along the whole section AB, i.e. with respect to time from t to $t + T$. Here we take into account the fact that the functions $\vartheta_i(-L, t)$ are periodic, so that

$$\vartheta_i(-L, t) = \vartheta_i(-L, t + T)$$

As a result we obtain

$$w_i^+(0, t + T/2) = w_i^+(-L, t) + F_i^+ T \tag{3.2}$$

The quantities F_i^+, obtained on integrating the right-hand sides of Eq. (2.4), were calculated taking into account the fact that on the right-hand boundary (at $x = 0$) the boundary condition of linear problem (2.6) must be satisfied. We have

$$F_i^+ = \frac{1}{4a^2}\left\{g_i - \frac{\kappa}{4a^2}[3(\varphi_i + \bar{\varphi}_i)^2 + (\varphi_{3-i} + \bar{\varphi}_{3-i})^2 + 3h_{ii} + h_{(3-i)(3-i)}]\right\}\frac{\partial\varphi_i}{\partial t} - \\ -\frac{\kappa}{8a^4}(\varphi_1 + \bar{\varphi}_1)(\varphi_2 + \bar{\varphi}_2) + h_{12})\frac{\partial\varphi_{3-i}}{\partial t} \tag{3.3}$$

where

$$\bar{\varphi}_i = \frac{2}{T}\int_0^T \varphi_i dt, \quad h_{ij} = \bar{\varphi}_{ij} - \bar{\varphi}_i\bar{\varphi}_j, \quad \bar{\varphi}_{ij} = \frac{2}{T}\int_0^T \varphi_i\varphi_j dt; \quad i, j = 1, 2 \tag{3.4}$$

The quantities defined by Eq. (3.4) may vary slowly from period to period, since for each step of the passage of the waves from one boundary to the other, the functions u_i obtained as a result of the passage of the preceding cycle act as the functions φ_i.

To calculate the functions $w_i^\pm$ on the left-hand boundary of the layer we can use the result of the integration of the second group of Eq. (2.4) along the characteristic of the second family DB from $t + T/2$ to $t + T$. The point $D(0, t + T/2)$ on the right-hand boundary of the layer serves as the initial point. The functions $b_i(\varphi_k, \vartheta_k)$, specified by Eq. (3.1), are integrated in the same way. Only the functions $\vartheta_i(at + x)$ and their derivatives will be constant on the characteristic BB, while the values of φ_i are taken along their characteristics ($x - at = \text{const}$) from the section AB of the left-hand boundary. As a result of the integration we obtain

$$w_i^-(-L, t + T) = w_i^-(0, /, t + T/2) + F_i^- T \tag{3.5}$$

Taking into account the fact that

$$(w_i^- + w_i^+)/2 = \upsilon_i \tag{3.6}$$

we obtain from the boundary condition at $x = 0$

$$w_i^-(0, t + T/2) = -w_i^+(0, t + T/2)$$

and from boundary condition (2.6) for the linear approximation it follows that

$$F_i^- = -F_i^+ = F_i, \quad \text{i.e.} \quad F_i^- - F_i^+ = 2F_i$$

As a result, using relation (3.2), Eq. (3.5) gives

$$w_i^-(-L, t + T) = -w_i^+(-L, t) + 2F_iT$$

4. The equations of the slow evolution of the waves

Adding and subtracting $w_i^-(-L, t)$ from the left-hand side of the last equation and taking into account, as above, Eq. (3.6), we can calculate the change in the function w_i^- on the left-hand boundary over the whole period T (from point A to point B in Fig. 1)

$$w_i^-(-L, t + T) - w_i^-(-L, t) = 2F_iT - 2\upsilon_i(-L, t) \tag{4.1}$$

Similarly we can calculate the change in w_i^+ over a period of the functions

$$w_i^+(-L, t + T) - w_i^+(-L, t) = -2F_iT + 2\upsilon_i(-L, t) \tag{4.2}$$

The last terms on the right-hand sides of Eqs. (4.1) and (4.2) are determined by the boundary conditions on the left-hand boundary, where $\upsilon_i(-L, t) = \psi_i(t)$ is specified.

Eqs. (4.1) and (4.2) describe the small change in the functions $w_i^\pm$ over a period due to the presence in Eq. (2.4) of the right-hand sides, due to the fact that non-linear effects and the external action have been taken into account when

$x=-L$. In addition to the real time t we can introduce into consideration a "slow" time τ for which the period T will be a small quantity. Then, dividing each of Eqs. (4.1) and (4.2) by T, we can represent them in the form of a system of partial differential equations, where one of the variables is the slow time τ and the other is the real time t, and the equations describe the change in the functions w_i on the left-hand boundary of the layer. We have

$$\frac{\partial w_i^-}{\partial \tau} = 2\left(F_i - \frac{\psi_i}{T}\right), \quad \frac{\partial w_i^+}{\partial \tau} = -2\left(F_i - \frac{\psi_i}{T}\right) \tag{4.3}$$

We can now return to the initial functions u_i and v_i, which occur in the formulation of the problem. According to relations (2.3) $w_i^- = v_i + au_i$, and on the left-hand boundary where $x=-L$

$$w_i^-(-L,t) = \psi_i(t) + au_i(-L,t) \tag{4.4}$$

Here we have assumed that ψ_i are periodic functions, i.e. they do not change in slow time: $\partial\psi_i/\partial\tau = 0$. Hence, the first equation of (4.3) can be represented in the form

$$\frac{\partial u_i}{\partial \tau} = \frac{2}{a}\left(F_i - \frac{\psi_i}{T}\right) \tag{4.5}$$

It was assumed when formulating the problem that the components of the deformation of the medium u_i are small, of the order of ε, and, consequently, the values of the functions φ_i are of the same order. As can be seen from expressions (3.3), the functions F_i are of the order of ε^3, and hence the value of the error will be less than ε^3, if we replace the functions φ_k in the expressions for F_i by u_i^0 using formulae (2.5) or approximately by the functions

$$\varphi_i(at) = au_i(-L,t)$$

Moreover, to obtain the system in the more usual form we replace the variable t by $\xi = at$. The variable ξ has the dimensions of length and, over a period of time T, varies from 0 to $2L$. As a result, system (4.5) takes the form

$$\frac{\partial u_i}{\partial \tau} + A_{ij}(u_k)\frac{\partial u_j}{\partial \xi} = -\frac{\psi_i(\xi)}{L}, \quad L = \frac{aT}{2}, \quad i,j,k = 1,2 \tag{4.6}$$

Here

$$\begin{aligned} A_{ii} &= \frac{1}{2a}\left\{g_i - \frac{\kappa}{4}[3(u_i+\bar{u}_i)^2 + (u_{3-i}+\bar{u}_{3-i})^2 + 3h_{ii} + h_{(3-i)(3-i)}]\right\} \\ A_{12} &= A_{21} = -\frac{\kappa}{4a}[(u_1+\bar{u}_1)(u_2+\bar{u}_2) + h_{12}] \\ \bar{u}_i &= \frac{2}{T}\int_0^T u_i dt, \quad h_{ij} = \bar{u}_{ij} - \bar{u}_i\bar{u}_j, \quad \bar{u}_{ij} = \frac{2}{T}\int_0^T u_i u_j dt \end{aligned} \tag{4.7}$$

Hence, we have obtained a system of two inhomogeneous differential equations in the variables τ, t with a symmetric matrix $||A_{ij}||$ for describing near-resonance oscillations in an elastic layer. In this system the change in τ is not limited, while t varies over a single period, while the end of one cycle is the beginning of the next one. Neglecting the change in quantities over a single cycle, we can assume that all the quantities are functions of t and τ, where t varies over a closed line of length T, while the dependence on τ describes the slow change of these functions. The solutions of Eqs. (4.6) and (4.7) give u_i for $x=-L$ and arbitrary values of the time. The quantity τ gives the number of the period considered while t indicates the time inside the period. Knowing $u_i(t)$ when $x=-L$ it is easy to obtain the values of u_i and v_i at any point x, t of the layer.

If we assume the quantities $\bar{u}_1$, $\bar{u}_2$, h_{ij} to be given, the system of Eqs. (4.6) and (4.7) are hyperbolic, which follows from the symmetry of the matrix $||A_{ij}||$. The elements of the matrix $||A_{ij}||$ can be represented in the form of second derivatives of a certain function $\mathcal{F}(u_1, u_2)$, which plays the role of the potential.

Eq. (4.6) take the form

$$\frac{\partial u_i}{\partial \tau} + \frac{\partial}{\partial \xi}\left(\frac{\partial \mathcal{F}}{\partial u_i}\right) = -\frac{\psi_i(\xi)}{L} \tag{4.8}$$

Here

$$\begin{aligned}\mathcal{F} &= \bar{f}\frac{u_1^2 + u_2^2}{2} + \bar{g}\frac{u_2^2 - u_1^2}{2} - \frac{\bar{\kappa}}{4}[(u_1 + \bar{u}_1)^2 + (u_2 + \bar{u}_2)^2]^2 - \bar{m}u_1u_2\\ \bar{f} &= \frac{f - a^2}{a} - 2\bar{\kappa}(h_{11} + h_{22}), \quad \bar{g} = \frac{g}{a} - \bar{\kappa}(h_{22} - h_{11}), \quad \bar{\kappa} = \frac{\kappa}{4a}, \quad \bar{m} = \frac{\kappa}{2a}h_{12}\end{aligned} \tag{4.9}$$

Obviously, f and g are of the order of ε^2, so that all the functions $\mathcal{F}$ are of the order of ε^4. The coefficients of the matrix $||A_{ij}||$ (4.7) are expressed in terms of the function $\mathcal{F}$ by the equalities

$$A_{ij} = \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial u_i \partial u_j}$$

Note that in the expressions for $\bar{f}$, $\bar{g}$, $\bar{\kappa}$, $\bar{m}$ the quantity a in the numerators of all the expressions can be replaced by the velocity of the linear isotropic waves $c_0 = \sqrt{f}$, while satisfying the assumed accuracy.

Expressions (4.7) for the coefficient A_{ij} and the potential $\mathcal{F}$ are simplified considerably when the functions u_i, when they are averaged over a period, have zero mean value $\bar{u}_i = 0$. It is easy to indicate the requirements which the functions of the external action ψ_i must satisfy in order to ensure the desired property of the solution. To do this it is sufficient to integrate Eq. (4.6) over a period, putting $\bar{u}_i = 0$ in them.

In any case, Eq. (4.8), which describe the slow change in the motion from period to period, have the same form as the well-investigated Eq. (1.3) for transverse waves, which propagate in one direction in an unbounded elastic medium, while the potential $\mathcal{F}$ has the same structure as the initial elastic potential Φ_1 in these equations. The absence of a term with the product u_1u_2 in the function Φ (1.1) was due to the special choice of the x_1 and x_2 axes in the plane of the wave front.

A considerable difference between system (4.8) and (1.3) is the fact that the averaged quantities $\bar{u}_i$, h_{ij} depend on τ, and consequently, the coefficients $\bar{f}$, $\bar{g}$ and $\bar{\kappa}$, which occur in expression (4.9) for the function $\mathcal{F}$, may vary slowly as τ increases. Nevertheless, we would expect that the solution of the problem of the evolution of the waves will have similar properties to the solution of the well-known problem of waves in an unbounded medium. In particular, the evolution of the solutions may lead to the formation of discontinuities.

5. A slow change in the period

We will now consider the case when a slow change occurs from period to period in the boundary conditions which specify the external action. We will also assume that the length of the period $T = T(\tau)$ and the coefficients of elasticity of the medium f, g and κ also vary slowly with time. It was pointed out earlier that since the external action is of the order of ε^3, and the perturbations themselves are of the order of ε, the characteristic time during which a considerable change in the oscillations can occur is of the order of ε^{-2}. If some changes in the boundary conditions occur after a considerably shorter time, we can assume that after the lapse of this time functions which define the oscillations are unable to change if the period of the external action T does not change, while in the case when T changes, the changes in all these functions can easily be taken into account in the linear approximation. If the characteristic time of variation of the boundary conditions is an order of magnitude greater than ε^{-2}, the oscillations can obviously be assumed to be quasi-stationary. Taking into account the fact that the difference in the period of the external action T from the time which is required by the characteristics to traverse the section L in both directions has an order of magnitude no less than ε^2, the quantity $\partial T/\partial t$ must be assumed to be of the order of ε^4.

The quantity $a = 2L/T$ in this case turns out to be variable, since T may slowly vary with time. Moreover, we will also assume that f, g and κ in expression (1.1) for Φ are slowly varying functions. Introducing the variable $t_1 = \int a(t)dt$

instead of t, we obtain the following new equations for the invariants (2.3) instead of system (2.4)

$$\begin{aligned} \frac{dw_i^+}{dt_1} &= \frac{1}{a(t_1)}b_i + u_i\frac{\partial a}{\partial t_1} \quad \text{along} \quad \frac{dx}{dt_1} = 1 \\ \frac{dw_i^-}{dt_1} &= \frac{1}{a(t_1)}b_i - u_i\frac{\partial a}{\partial t_1} \quad \text{along} \quad \frac{dx}{dt_1} = -1 \end{aligned} \tag{5.1}$$

According to the estimate of the quantity $\partial T/\partial t$ made above, the terms with $\partial a/\partial t_1$ in the cases of interest which occur are of the order of ε^5 in Eq. (5.1) and can be dropped. If we assume that the change in the coefficients f, g and κ are related to the change in entropy, which occurs during the transmission of shock waves, the amplitude of which is of the order of ε, then after a single period the change in the coefficients will not be of an order of magnitude exceeding ε^4.[2] If we confine ourselves to this case, then, when calculating the change in w_i^+ and w_i^- over a period we can ignore the changes in the quantities a, f, g and κ over one period, and only take into account the average changes of these quantities over many periods, i.e. we can assume that a, f, g and κ are slowly varying functions of time, i.e. they are functions of τ. Further calculations repeat the case when $T = \text{const}$, with the sole difference that the functions of the zeroth approximation φ_i and ϑ_i will now depend on $t_1 - x$ and $t_1 + x$ respectively. Here expressions (3.1) for b_1 and b_2 will be retained with $\partial/\partial t$ replaced by $a(\tau)\partial/\partial t_1$, and also the final Eqs. (4.6) and (4.7), taking into account the fact that the coefficients occurring in (4.7) and the functions ψ_i depend both on the real time t and on the slow time τ.

6. Maintaining a steady periodic mode of oscillations in the layer

Eq. (4.6) enable us to consider the problem when it is required to maintain or, in a certain way, change the oscillations of an elastic layer. We will consider the problem of maintaining periodic oscillations in the case when the oscillations are steady. Thus, we will assume that

$$\frac{\partial u_i}{\partial \tau} = 0, \quad u_i(t) = u_i(t+T), \quad T, \bar{u}_i, h_{ij} \text{ – are constant quantities}$$

Then, for a specified desired form of the solution $u_i(t)$, the remaining terms enable us to obtain functions of the external action $\psi_i(t)$, which in each cycle will compensate the non-linear effects that occur. We have

$$\psi_i(t) = -A_{ij}\frac{L}{a}\frac{du_j(t)}{dt}$$

It can be seen from these equations and expressions (4.7) for A_{ij}, that the functions $\psi_i(t)$, which define the external actions, are of the order of ε^3, if the main solution u_i is of the order of ε.

To illustrate this problem we will take a simple example. Suppose the periodic solution, which must be maintained in the layer, has the following form on the boundary

$$u_1 = A\sin\omega t, \quad u_2 = B\cos\omega t$$

Here $\omega = 2\pi/T$, while the period T is close to the resonance period $T_0 = 2L/c_0$, defined by the velocities of the transverse waves in the elastic material considered. Then

$$\bar{u}_1 = \bar{u}_2 = h_{12} = 0, \quad h_1 = A^2, \quad h_2 = B^2$$

and we obtain the following expressions for the functions ψ_i

$$\psi_1 = -\frac{A\pi}{2a^2}\left[g_1 - \frac{\kappa}{8}(9A^2 + B^2)\right]\cos\omega t - \frac{3A\pi\kappa}{32a^2}(A^2 - B^2)\left(\cos\frac{\omega t}{2} + \cos\frac{3\omega t}{2}\right)$$

$$\psi_2 = \frac{B\pi}{2a^2}\left[g_2 - \frac{\kappa}{8}(A^2 + 9B^2)\right]\sin\omega t - \frac{3B\pi\kappa}{32a^2}(A^2 - B^2)\left(\sin\frac{\omega t}{2} - \sin\frac{3\omega t}{2}\right)$$

which consists of the sum of several harmonics with periods that are a multiple of the specified period. The first terms in these expressions recall the result of the investigation of the linear problem. For certain values of a (defined by the period T) one of these terms may vanish. The second terms in the expressions for ψ_i vanish when $A = B$.

Acknowledgements

This research was supported financially by the Russian Foundation for Basic Research (05-01-00219, 05-01-00375) and the Programme for the Support for the Leading Scientific Schools (NSh-4710.2006.1).

References

1. Bland DR. *Nonlinear Dynamic Elasticity*. Toronto: Waltham; 1969.
2. Kulikovskii AG, Sveshnikova EI. *Non-linear Waves in Elastic Media*. Moscow: Mosk. Litsei; 1998.
3. Chester W. Resonant oscillations in closed tubes. *J Fluid Mech* 1964;**18**(1):44–64.
4. Kraiko AN, Ni AL. The approximation of non-linear acoustics in problems of the oscillations of a gas in tubes. *Prikl Mat Mekh* 1980;**44**(1):77–88.
5. Ni AL. Non-linear resonance oscillations of a gas in a tube under the action of a periodically varying pressure. *Prikl Mat Mekh* 1983;**47**(4):607–18.
6. Gusev VE. The establishment of forced oscillations in acoustic resonators. *Akust Zh* 1984;**30**(2):204–12.
7. Yegorushkin SA. Near-resonance oscillations of a gas moving in a channel of variable cross section. *Izv Akad Nauk SSSR MZhG* 1984;**4**:107–15.
8. Enflo BO, Hedberg CM, Rudenko O. Nonlinear standing waves in a layer excited by the periodic motion of its boundary. *Proc. 16th Intern. Symp. Nonlinear Acoustics. Nonlinear Acoustics at Beginning of the 21st Century*. Moscow: Faculty of Physics Moscow State Univ., 2002;1:49–52.
9. Sibgatullin NR. Non-linear transverse oscillations at resonance in an elastic layer and a layer of ideally conducting liquid. *Prikl Mat Mekh* 1972;**36**(1):79–87.
10. Sibgatullin NR. The non-linear mechanism of exciting shock waves in bounded systems for resonance of transverse oscillations. *Izv Akad Nauk SSSR MZhG* 1976;**2**:131–9.
11. Kulikovskii AG. The equations describing the propagation of non-linear quasitransverse waves in a slightly non-isotropic elastic body. *Prikl Mat Mekh* 1986;**50**(4):597–604.

Translated by R.C.G.